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Pure spinor formalism as an $N = 2$ topological string

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Abstract: Following suggestions of Nekrasov and Siegel, a non-minimal set of fields are added to the pure spinor formalism for the superstring. Twisted $\hat{c} = 3$ $N = 2$ generators are then constructed where the pure spinor BRST operator is the fermionic spin-one generator, and the formalism is interpreted as a critical topological string. Three applications of this topological string theory include the super-Poincaré covariant computation of multiloop superstring amplitudes without picture-changing operators, the construction of a cubic open superstring field theory without contact-term problems, and a new four-dimensional version of the pure spinor formalism which computes F-terms in the spacetime action.

Keywords: Superstrings and Heterotic Strings, Topological Strings.

Contents

1. Introduction

Five years ago, a new formalism for the superstring was proposed which is manifestly super-Poincaré covariant and which can be easily quantized [1, 2]. The main new feature of the formalism is a BRST operator $Q = \int dz \lambda^\alpha d_\alpha$ constructed from the fermionic Green-Schwarz constraint d_α and a bosonic ghost λ^α satisfying the pure spinor constraint $\lambda^\alpha \gamma^m_{\alpha\beta} \lambda^\beta = 0$. This super-Poincaré covariant formalism has had various applications such as quantization of the superstring in an $AdS_5 \times S^5$ Ramond-Ramond background [3] and computation of multiloop scattering amplitudes [4].

Because of the simple but unconventional form of the BRST operator, it is not obvious how it can be obtained by gauge-fixing a reparameterization-invariant worldsheet action. Although the matter sector of the formalism involves the standard Green-Schwarz-Siegel worldsheet variables, the ghost sector is lacking the usual (b, c) ghosts and involves a constrained bosonic ghost λ^α with ghost-number anomaly -8 whose complex conjugate is absent from the formalism. In this paper, these mysterious features of the pure spinor formalism will be explained.

Following suggestions of Nekrasov and Siegel, a non-minimal set of variables which include the complex conjugate to λ^α and a fermionic constrained spinor are added to the pure spinor formalism. These non-minimal variables do not affect the BRST cohomology but change the ghost-number anomaly from -8 to $+3$. The new variables are closely related to the variables used for (β, γ) systems in the N=(0,2) models discussed in [5]. A twisted set of $\hat{c} = 3$ $N = 2$ superconformal generators are then constructed out of the non-minimal variables such that the pure spinor BRST operator is the fermionic spin-one generator. This $\hat{c} = 3$ $N = 2$ superconformal field theory is then interpreted as a critical topological string [6, 7] in which the fermionic spin-two generator plays the role of the b ghost.

In this topological string interpretation of the pure spinor formalism, the simple form of the BRST operator and the absence of fundamental (b, c) ghosts are naturally explained. Furthermore, it will be possible to apply standard topological string methods to compute super-Poincaré covariant multiloop superstring amplitudes, construct a cubic superstring field theory action, and compactify the pure spinor formalism to four dimensions.

Using the old "minimal" version of the pure spinor formalism, a multiloop amplitude prescription involving picture-changing operators was proposed in [4]. Because the picture-changing operators required choices of constant spacetime spinors and tensors, this prescription was only Lorentz-covariant up to BRST-trivial surface terms. Using the new "non-minimal" version of the pure spinor formalism, multiloop superstring amplitudes can now be computed using topological string methods in which the picture-changing operators are replaced by a regularization factor for the zero modes. This "non-minimal" prescription is manifestly Lorentz-covariant and is expected to reproduce the "minimal" prescription in a gauge in which the contribution from the non-minimal fields decouple.

Since the superstring amplitude prescription no longer requires picture-changing operators, the analogous open superstring field theory action does not require singular insertions at the midpoint. Using standard topological methods, one can therefore construct a cubic open superstring field theory action resembling the Chern-Simons action [6] which does not suffer from contact-term or gauge invariance problems. Construction of a similar action was attempted four years ago by Schwarz and Witten [8], but was abandoned because of difficulties caused by the "minimal" pure spinor measure factor. It would be interesting to generalize this construction to a closed superstring field theory action which might resemble the Kodaira-Spencer action [7].

Critical topological strings describe Calabi-Yau compactifications to four dimensions [7, 9], so it is natural to consider a four-dimensional version of the pure spinor formalism in which λ^a is a $d = 4$ pure spinor, i.e. a two-component chiral spinor. After including the $(x^m, \theta^a, \overline{\theta}^{\dot{a}}, p_a, \overline{p}_{\dot{a}})$ variables of N=1 $d = 4$ superspace, as well as the appropriate non-minimal variables, one finds that the $d = 4$ version of the pure spinor formalism has $\hat{c} = 0$. So after adding an $N = 2$ $\hat{c} = 3$ sector for the Calabi-Yau variables, one obtains a critical topological string with manifest $d = 4$ super-Poincaré invariance. But unlike the $d = 4$ hybrid formalism [10] which is related to the RNS formalism by a field redefinition and describes the complete superstring, this new formalism only describes the chiral sector of $d = 4$ superstring theory. Note that unlike in $d = 10$, $Q = \lambda^a d_a$ has trivial cohomology in $d = 4$, so the four-dimensional pure spinor formalism cannot be used to compute generic superstring amplitudes. Nevertheless, the formalism can be used to compute F-terms in the spacetime action, and can be understood as a $d = 4$ super-Poincaré covariant version of the $\hat{c} = 5$ topological string introduced in [11]. Hopefully, this new four-dimensional formalism will be useful for studying the effect of Ramond-Ramond fields on the spacetime superpotential.

In earlier papers, there have been various proposals for a more "geometric" version of the pure spinor formalism, some of which share certain properties with the non-minimal pure spinor formalism presented here. For example, one proposal suggests relaxing the pure spinor constraint and adding ghosts-for-ghosts to the formalism which allows $N = 2$ worldsheet supersymmetry [13]. However, the $N = 2$ worldsheet supersymmetry transformations in this proposal are quite different from the $N = 2$ transformations in the non-minimal pure spinor formalism, and the ghosts-for-ghosts do not play the role of non-minimal fields since they affect the BRST cohomology.

Another proposal has been to obtain the pure spinor formalism from an extended Green-Schwarz formalism which involves an additional fermionic spinor variable [14 – 16]. Unfortunately, the pure spinor BRST operator is obtained in this proposal by passing through a complicated procedure which has up to now only been defined in semi-light-cone gauge. Since the structure of the worldsheet ghosts and supermoduli in semi-light-cone gauge is not well understood, this proposal has not yet shed much light on the pure spinor formalism. Nevertheless, it is interesting that the non-minimal pure spinor formalism also involves an additional fermionic spinor variable.

A third proposal has been to relate the pure spinor formalism to an $N = 2$ superembedding of the Green-Schwarz superstring [17], also known as the $N = 2$ twistor-string [18], and to the $d = 4$ hybrid formalism [19]. Although the $N = 2$ twistor-string has only been covariantly studied at the classical level, it can be quantized in a U(4)-covariant manner [20] and related to the hybrid formalism for the superstring which has $\hat{c} = 2$ [21, 22]. Despite the fact that the $N = 2$ twistor-string and hybrid formalism have different central charge from the non-minimal pure spinor formalism, the classical $N = 2$ worldsheet supersymmetry transformations are very similar in the formalisms. It would be very interesting to understand the relation between the $\hat{c} = 3$ non-minimal pure spinor formalism which describes a critical topological $N = 2$ string and the $\hat{c} = 2$ hybrid formalism which describes a critical non-topological $N = 2$ string.

There have also been papers which expand on the analogy with Chern-Simons in [2, 23] to find various topological properties of the pure spinor formalism [24 – 27]. These topological properties include the construction of the Batalin-Vilkovisky action, the role of the pure spinor measure factor, and the geometrical interpretation of picture-changing operators in amplitude computations.

Finally, there have been versions of the pure spinor formalism which involve additional fields such as the Y-formalism [28] and a pure spinor version [29] of the "Big Picture" formalism [30]. Although the additional fields in these two approaches share some properties with the non-minimal fields used here, it is the $N = (0, 2)$ model proposed by Nekrasov [31] for the $(\lambda^\alpha, w_\alpha)$ ghosts of the pure spinor formalism which most closely resembles the non-minimal formalism of this paper.

In section 2 of this paper, the "minimal" pure spinor formalism will be reviewed. In section 3, a set of "non-minimal" variables will be added to the formalism and twisted $\hat{c} = 3$ $N = 2$ generators will be constructed. In section 4, this critical topological string wil be used to compute superstring scattering amplitudes up to two loops. In section 5, a consistent cubic open superstring field theory action will be constructed. In section 6, a new four-dimensional version of the pure spinor formalism will be defined which computes F-terms in the spacetime action. And in the appendix, the constrained variables of the non-minimal pure spinor formalism will be solved in terms of U(5)-covariant free fields.

2. Review of minimal pure spinor formalism

2.1 Worldsheet variables

As in Siegel's approach to the Green-Schwarz superstring [32], the pure spinor formalism for the superstring is constructed using the (x^m, θ^α) variables of $d = 10$ superspace where $m =$ 0 to 9 and $\alpha = 1$ to 16, together with the fermionic conjugate momenta p_α. Furthermore, one introduces a bosonic spinor ghost λ^α which satisfies the pure spinor constraint

$$\lambda^\alpha \gamma^m_{\alpha\beta} \lambda^\beta = 0\,, \tag{2.1}$$

where $\gamma^m_{\alpha\beta}$ are the symmetric 16×16 $d = 10$ Pauli matrices.

Because of the pure spinor constraint on λ^α, its conjugate momentum w_α is defined up to the gauge transformation

$$\delta w_\alpha = \Lambda^m (\gamma_m \lambda)_\alpha\,, \tag{2.2}$$

which implies that w_α only appears through its Lorentz current N_{mn}, ghost current J_λ, and stress tensor T_λ. These gauge-invariant currents are defined by

$$N_{mn} = \frac{1}{2} w \gamma_{mn} \lambda\,, \qquad J_\lambda = w_\alpha \lambda^\alpha\,, \qquad T_\lambda = w_\alpha \partial \lambda^\alpha\,. \tag{2.3}$$

The worldsheet action for the left-moving matter and ghost variables is

$$S = \int d^2 z \left(\frac{1}{2} \partial x^m \overline{\partial} x_m + p_\alpha \overline{\partial} \theta^\alpha - w_\alpha \overline{\partial} \lambda^\alpha \right), \tag{2.4}$$

and the right-moving variables will be ignored throughout this paper. For the Type II superstring, the right-moving variables are similar to the left-moving variables, while for the heterotic superstring, the right-moving variables are the same as in the RNS heterotic formalism.

The OPE's for the matter variables are easily computed to be

$$x^m(y) x^n(z) \to -\eta^{mn} \log |y - z|^2\,, \qquad p_\alpha(y) \theta^\beta(z) \to \delta^\beta_\alpha (y - z)^{-1}\,, \tag{2.5}$$

however, the pure spinor constraint on λ^α prevents a direct computation of the OPE's for the ghost variables. Nevertheless, one can compute OPE's involving λ^α and the currents of (2.3) either by solving the pure spinor constraint in terms of U(5)-covariant free fields [1], by using the SO(10)-covariant fixed-point techniques of [33], or by using the Y-formalism of [28]. The resulting OPE's are

$$\begin{aligned}
N_{mn}(y)\lambda^\alpha(z) &\to \frac{1}{2}(y-z)^{-1}(\gamma_{mn}\lambda)^\alpha\,, \qquad J(y)\lambda^\alpha(z) \to (y-z)^{-1}\lambda^\alpha\,,\\
N^{kl}(y)N^{mn}(z) &\to -3(y-z)^{-2}\left(\eta^{n[k}\eta^{l]m}\right) + (y-z)^{-1}\left(\eta^{m[l}N^{k]n} - \eta^{n[l}N^{k]m}\right),\\
J_\lambda(y)J_\lambda(z) &\to -4(y-z)^{-2}\,, \qquad J_\lambda(y)N^{mn}(z) \to \text{regular}\,,\\
N_{mn}(y)T_\lambda(z) &\to (y-z)^{-2}N_{mn}(z)\,, \qquad J_\lambda(y)T_\lambda(z) \to -8(y-z)^{-3} + (y-z)^{-2}J_\lambda(z)\,,\\
T_\lambda(y)T_\lambda(z) &\to 11(y-z)^{-4} + 2(y-z)^{-2}T_\lambda(z) + (y-z)^{-1}\partial T_\lambda(z)\,. \qquad (2.6)
\end{aligned}$$

From the above OPE's, one sees that the central charge contribution to the conformal anomaly is 22, the level for the Lorentz currents is -3, and the ghost-number anomaly is -8. So the central charge contribution from the ghost variables cancels the contribution of $+10-32=-22$ from the $(x^m, \theta^\alpha, p_\alpha)$ matter variables. Furthermore, the total Lorentz current is $M_{mn} = -\frac{1}{2}(p\gamma_{mn}\theta) + N_{mn}$, and since $-\frac{1}{2}(p\gamma_{mn}\theta)$ has level $+4$, M_{mn} has the same level of $+4-3=1$ as the RNS Lorentz current $M_{mn} = \psi_m\psi_n$. Finally, it will be explained in the following section that after adding a set of non-minimal variables, the ghost-number anomaly of -8 is shifted to the usual ghost-number anomaly of $+3$.

2.2 Physical states

Physical open string states in the pure spinor formalism are defined as ghost-number one states in the cohomology of the nilpotent BRST operator

$$Q = \int dz\ \lambda^\alpha d_\alpha\,, \qquad (2.7)$$

where

$$d_\alpha = p_\alpha - \frac{1}{2}\gamma^m_{\alpha\beta}\theta^\beta\partial x_m - \frac{1}{8}\gamma^m_{\alpha\beta}\gamma_{m\gamma\delta}\theta^\beta\theta^\gamma\partial\theta^\delta \qquad (2.8)$$

is the supersymmetric Green-Schwarz constraint. As shown by Siegel [32], d_α satisfies the OPE's

$$\begin{aligned}
d_\alpha(y)d_\beta(z) &\to -(y-z)^{-1}\gamma^m_{\alpha\beta}\Pi_m\,, \qquad d_\alpha(y)\Pi^m(z) \to (y-z)^{-1}\gamma^m_{\alpha\beta}\partial\theta^\beta(z)\,,\\
d_\alpha(y)\ f(x(z),\theta(z)) &\to (y-z)^{-1}D_\alpha f(x(z),\theta(z))\,, \qquad (2.9)
\end{aligned}$$

where

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + \frac{1}{2}\theta^\beta\gamma^m_{\alpha\beta}\partial_m \qquad (2.10)$$

is the $d=10$ supersymmetric derivative, $\Pi^m = \partial x^m + \frac{1}{2}\theta\gamma^m\partial\theta$ is the supersymmetric momentum and

$$q_\alpha = \int dz\left(p_\alpha + \frac{1}{2}\gamma^m_{\alpha\beta}\theta^\beta\partial x_m + \frac{1}{24}\gamma^m_{\alpha\beta}\gamma_{m\ \gamma\delta}\theta^\beta\theta^\gamma\partial\theta^\delta\right) \qquad (2.11)$$

is the supersymmetric generator satisfying

$$\{q_\alpha, q_\beta\} = \gamma^m_{\alpha\beta}\int dz\partial x_m\,, \qquad [q_\alpha, \Pi^m(z)] = 0\,, \qquad \{q_\alpha, d_\beta(z)\} = 0\,. \qquad (2.12)$$

For massless states described by $V = \lambda^\alpha A_\alpha(x,\theta)$, $QV = 0$ and $\delta V = Q\Omega$ implies that A_α is the super-Yang-Mills spinor gauge field satisfying the linearized equation of motion $(\gamma_{mnpqr})^{\alpha\beta} D_\alpha A_\beta = 0$ and the linearized gauge invariance $\delta A_\alpha = D_\alpha \Omega$. For massive states, the superspace description is more complicated [34], however, it has been proven by DDF methods that the cohomology of Q at ghost-number one correctly describes the open superstring spectrum [35].

2.3 Scattering amplitudes

To compute scattering amplitudes using the "minimal" pure spinor formalism, it is necessary to introduce picture-changing operators which can absorb the zero modes of the bosonic ghosts λ^α and w_α. For example, N-point tree amplitudes are computed by the correlation function

$$\mathcal{A} = \left\langle V_1(z_1)V_2(z_2)V_3(z_3) \int dz_4 U_4(z_4) \cdots \int dz_N U_N(z_N) \prod_{I=1}^{11} Y_{C_I}(y_I) \right\rangle, \tag{2.13}$$

where $Y_{C_I} = C_{I\alpha}\theta^\alpha \delta(C_{I\beta}\lambda^\beta)$ are picture-lowering operators which absorb the eleven λ^α zero modes, $C_{I\alpha}$ are constant spinors, and U_r are dimension-one vertex operators which are related to the unintegrated vertex operators V_r by the relation $QU_r = \partial V_r$. This tree amplitude prescription has been shown to coincide with the RNS prescription for massless states with an arbitrary number of bosons and up to four fermions [36].

N-point g-loop amplitudes can also be computed in the minimal pure spinor formalism by evaluating the correlation function

$$\begin{aligned} \mathcal{A} = \int d^{3g-3}\tau \Big\langle \prod_{j=1}^{3g-3} \left(\int dw_j \mu_j(w_j) \tilde{b}_{B_j}(w_j) \right) \prod_{P=3g-2}^{10g} Z_{B_P}(w_P) \prod_{R=1}^{g} Z_J(v_R) \times \\ \times \prod_{I=1}^{11} Y_{C_I}(y_I) \prod_{r=1}^{N} \int dz_r \, \mathrm{U}(z_r) \Big\rangle, \end{aligned} \tag{2.14}$$

where τ_j are complex Teichmuller parameters and μ_j are the associated Beltrami differentials, $Z_B = B_{mn}(\lambda\gamma^{mn}d)\delta(B_{mn}N^{mn})$ and $Z_J = (\lambda^\alpha d_\alpha)\delta(J_\lambda)$ are picture-raising operators which absorb the $11g$ zero modes of w_α, B_{mn} are constant tensors, and $\tilde{b}_B$ is a picture-raised b ghost which is defined to satisfy $\{Q, \tilde{b}_B\} = TZ_B$. Although the explicit form of $\tilde{b}_B$ is quite complicated, this amplitude prescription has been used to prove various vanishing theorems and to compute four-point one-loop and two-loop massless amplitudes [4, 37, 38].

Although the choices of constant spinors C_α and tensors B_{mn} in the picture-changing operators Y_C and Z_B break manifest Lorentz covariance, one can show that the dependence on C_α and B_{mn} is BRST-trivial. So after integrating over the Teichmuller parameters, the scattering amplitude is independent of the choices for C_α and B_{mn}. Nevertheless, it would be more convenient if Lorentz covariance could be manifestly preserved at all stages in the amplitude computation. As will now be shown, this is possible using a "non-minimal" version of the pure spinor formalism in which picture-changing operators are replaced by a regularization factor for the zero modes.

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3. Non-minimal pure spinor formalism

3.1 Worldsheet variables

Although the BRST operator in the pure spinor formalism has a simple structure, the lack of a geometrical interpretation of the formalism makes it difficult to understand the rules for computing scattering amplitudes. As will be explained here, after introducing a set of non-minimal variables, the pure spinor formalism can be interpreted as a critical topological string with the standard topological rules for computing scattering amplitudes.

The new non-minimal variables will consist of a bosonic pure spinor $\overline{\lambda}_\alpha$ and a constrained fermionic spinor r_α satisfying the constraints

$$\overline{\lambda}_\alpha \gamma_m^{\alpha\beta} \overline{\lambda}_\beta = 0 \qquad \text{and} \qquad \overline{\lambda}_\alpha \gamma_m^{\alpha\beta} r_\beta = 0\,. \tag{3.1}$$

In d=10 euclidean space where complex conjugation flips the chirality of spacetime spinors, $\overline{\lambda}_\alpha$ can be interpreted as the complex conjugate to λ^α. The worldsheet action for the non-minimal pure spinor formalism is

$$\int d^2 z \left(\frac{1}{2} \partial x^m \overline{\partial} x_m + p_\alpha \overline{\partial} \theta^\alpha - w_\alpha \overline{\partial} \lambda^\alpha - \overline{w}^\alpha \overline{\partial} \overline{\lambda}_\alpha + s^\alpha \overline{\partial} r_\alpha \right), \tag{3.2}$$

where $\overline{w}^\alpha$ and s^α are the conjugate momenta for $\overline{\lambda}_\alpha$ and r_α with $+1$ conformal weight. As explained in the appendix, the constraints of (3.1) can be solved in a U(5)-covariant manner and $\overline{\lambda}_\alpha$ and r_α can be expressed in terms of eleven independent bosonic and fermionic free fields. Note that all non-minimal variables are left-moving on the worldsheet (like λ^α and θ^α), and that $\overline{\lambda}_\alpha$ and r_α are spacetime spinors of opposite chirality from λ^α and θ^α. It is interesting that similar variables to $\overline{\lambda}_\alpha$ and r_α have recently been used in N=(0,2) models for chiral (β, γ) systems [5]. However, unlike in these N=(0,2) models where the additional variables move in the opposite direction on the worldsheet from the (β, γ) variables, the non-minimal variables in the pure spinor formalism move in the same direction on the worldsheet as the $(\lambda^\alpha, w_\alpha)$ variables.

Just as w_α can only appear in the gauge-invariant combinations

$$N_{mn} = \frac{1}{2}(w \gamma_{mn} \lambda)\,, \qquad J_\lambda = w_\alpha \lambda^\alpha\,, \qquad T_\lambda = w_\alpha \partial \lambda^\alpha\,, \tag{3.3}$$

the variables $\overline{w}^\alpha$ and s^α can only appear in the combinations

$$\begin{aligned} \overline{N}_{mn} &= \frac{1}{2}(\overline{w} \gamma_{mn} \overline{\lambda} - s \gamma_{mn} r)\,, \qquad J_{\overline{\lambda}} = \overline{w}^\alpha \overline{\lambda}_\alpha - s^\alpha r_\alpha\,, \qquad T_{\overline{\lambda}} = \overline{w}^\alpha \partial \overline{\lambda}_\alpha - s^\alpha \partial r_\alpha\,, \\ S_{mn} &= \frac{1}{2} s \gamma_{mn} \overline{\lambda}, \quad S = s^\alpha \overline{\lambda}_\alpha\,, \end{aligned} \tag{3.4}$$

which are invariant under the gauge transformations

$$\delta \overline{w}^\alpha = \overline{\Lambda}^m (\gamma_m \overline{\lambda})^\alpha - \phi^m (\gamma_m r)^\alpha\,, \qquad \delta s^\alpha = \phi^m (\gamma_m \overline{\lambda})^\alpha \tag{3.5}$$

for arbitrary $\overline{\Lambda}^m$ and ϕ^m. Note that $J_r = r_\alpha s^\alpha$ and $\Phi = \overline{w}^\alpha r_\alpha$ are also gauge-invariant, but they can be written in terms of the other currents as

$$J_r = \frac{(\lambda r) S - \frac{2}{3}(\lambda \gamma^{mn} r) S_{mn}}{(\lambda \overline{\lambda})}, \quad \Phi = \frac{(\lambda r)(\overline{J}_{\overline{\lambda}} + J_r) - \frac{2}{3}(\lambda \gamma^{mn} r) \overline{N}_{mn}}{(\lambda \overline{\lambda})}\,. \tag{3.6}$$

These gauge-invariant currents will be shown in the appendix to satisfy the OPE's

$$
\begin{aligned}
\overline{N}_{mn}(y)\overline{\lambda}_\alpha(z) &\to \frac{1}{2}(y-z)^{-1}(\gamma_{mn}\overline{\lambda})_\alpha\,, \qquad \overline{N}_{mn}(y)r_\alpha(z) \to \frac{1}{2}(y-z)^{-1}(\gamma_{mn}r)_\alpha\,,\\
\overline{J}_{\overline{\lambda}}(y)\overline{\lambda}_\alpha(z) &\to (y-z)^{-1}\overline{\lambda}_\alpha\,, \qquad J_{\overline{\lambda}}(y)r_\alpha(z) \to (y-z)^{-1}r_\alpha\,,\\
\overline{N}^{kl}(y)\overline{N}^{mn}(z) &\to (y-z)^{-1}(\eta^{m[l}\overline{N}^{k]n} - \eta^{n[l}\overline{N}^{k]m})\,,\\
\overline{J}_{\overline{\lambda}}(y)\overline{N}^{mn}(z) &\to \text{regular}\,, \qquad J_r(y)\overline{N}^{mn}(z) \to \text{regular}\,, \qquad \Phi(y)\overline{N}^{mn}(z) \to \text{regular}\,,\\
\Phi(y)\overline{\lambda}_\alpha(z) &\to (y-z)^{-1}r_\alpha\,,\ \Phi(y)S_{mn}(z) \to (y-z)^{-1}\overline{N}_{mn}\,,\ \Phi(y)S(z) \to (y-z)^{-1}\overline{J}_{\overline{\lambda}}\,,\\
\overline{J}_{\overline{\lambda}}(y)\overline{J}_{\overline{\lambda}}(z) &\to \text{regular}\,, \qquad J_r(y)J_r(z) \to 11(y-z)^{-2}\,, \qquad J_{\overline{\lambda}}(y)J_r(z) \to 8(y-z)^{-2}\,,\\
\overline{N}_{mn}(y)T_{\overline{\lambda}}(z) &\to (y-z)^{-2}\overline{N}_{mn}(z)\,,\\
\overline{J}_{\overline{\lambda}}(y)T_{\overline{\lambda}}(z) &\to (y-z)^{-2}\overline{J}_{\overline{\lambda}}(z)\,,\\
J_r(y)T_{\overline{\lambda}}(z) &\to 11(y-z)^{-3} + (y-z)^{-2}J_r(z)\,,\\
T_{\overline{\lambda}}(y)T_{\overline{\lambda}}(z) &\to 2(y-z)^{-2}T_{\overline{\lambda}}(z) + (y-z)^{-1}\partial T_{\overline{\lambda}}(z)\,.
\end{aligned}
\tag{3.7}
$$

From the above OPE's, one sees that the non-minimal variables do not contribute to the conformal anomaly or to the level of the Lorentz currents. Furthermore, if the ghost current is defined as $w_\alpha\lambda^\alpha - \overline{w}^\alpha\overline{\lambda}_\alpha = J_\lambda - \overline{J}_{\overline{\lambda}} + J_r$, the non-minimal variables shift the ghost-number anomaly to $-8+11=+3$, which is the same ghost-number anomaly as in bosonic string theory.

3.2 $\hat{c}=3$ $N=2$ generators

In order that the non-minimal variables do not affect the cohomology, the "minimal" pure spinor BRST operator $Q=\int dz\lambda^\alpha d_\alpha$ will be modified to the "non-minimal" BRST operator [31]

$$
Q_{\text{nonmin}} = \int dz(\lambda^\alpha d_\alpha + \overline{w}^\alpha r_\alpha)\,. \tag{3.8}
$$

The new term $\int dz\overline{w}^\alpha r_\alpha$ is invariant under the gauge transformation of (3.5) and implies through the usual quartet argument that the cohomology is independent of $(\overline{\lambda}_\alpha, \overline{w}^\alpha)$ and (r_α, s^α).

In the "minimal" pure spinor formalism, one could have defined a non-covariant b ghost satisfying $\{Q,b\}=T$ as [12]

$$
b = \frac{C_\alpha G^\alpha}{C_\alpha\lambda^\alpha}\,, \tag{3.9}
$$

where C_α is any constant spinor and

$$
G^\alpha = \frac{1}{2}\Pi^m(\gamma_m d)^\alpha - \frac{1}{4}N_{mn}(\gamma^{mn}\partial\theta)^\alpha - \frac{1}{4}J_\lambda\partial\theta^\alpha - \frac{1}{4}\partial^2\theta^\alpha \tag{3.10}
$$

satisfies $\{Q,G^\alpha\} = \lambda^\alpha T$. However, such a b ghost contains poles when $C_\alpha\lambda^\alpha=0$, which causes problems in the presence of picture-changing operators containing factors of $\delta(\lambda)$.

In the non-minimal pure spinor formalism, there will be no picture-changing operators and one can define a Lorentz-invariant b_{nonmin} ghost satisfying $\{Q_{\text{nonmin}}, b_{\text{nonmin}}\} =$

$T_{\rm nonmin}$ as

$$\begin{aligned} b_{\rm nonmin} &= s^\alpha \partial\overline{\lambda}_\alpha + \frac{\overline{\lambda}_\alpha G^\alpha}{(\overline{\lambda}\lambda)} + \frac{\overline{\lambda}_\alpha r_\beta H^{[\alpha\beta]}}{(\overline{\lambda}\lambda)^2} - \frac{\overline{\lambda}_\alpha r_\beta r_\gamma K^{[\alpha\beta\gamma]}}{(\overline{\lambda}\lambda)^3} - \frac{\overline{\lambda}_\alpha r_\beta r_\gamma r_\delta L^{[\alpha\beta\gamma\delta]}}{(\overline{\lambda}\lambda)^4} \\ &= s^\alpha \partial\overline{\lambda}_\alpha + \frac{\overline{\lambda}_\alpha (2\Pi^m(\gamma_m d)^\alpha - N_{mn}(\gamma^{mn}\partial\theta)^\alpha - J_\lambda \partial\theta^\alpha - \frac{1}{4}\partial^2\theta^\alpha)}{4(\overline{\lambda}\lambda)} + \\ &\quad + \frac{(\overline{\lambda}\gamma^{mnp}r)(d\gamma_{mnp}d + 24 N_{mn}\Pi_p)}{192(\overline{\lambda}\lambda)^2} - \frac{(r\gamma_{mnp}r)(\overline{\lambda}\gamma^m d)N^{np}}{16(\overline{\lambda}\lambda)^3} + \\ &\quad + \frac{(r\gamma_{mnp}r)(\overline{\lambda}\gamma^{pqr}r)N^{mn}N_{qr}}{128(\overline{\lambda}\lambda)^4}, \end{aligned} \tag{3.11}$$

where

$$T_{\rm nonmin} = -\frac{1}{2}\partial x^m \partial x_m - p_\alpha \partial\theta^\alpha + w_\alpha \partial\lambda^\alpha + \overline{w}^\alpha \partial\overline{\lambda}_\alpha - s^\alpha \partial r_\alpha\,, \tag{3.12}$$

and $(G^\alpha, H^{\alpha\beta}, K^{\alpha\beta\gamma}, L^{\alpha\beta\gamma\delta})$ are operators which were defined in [4, 28] for constructing the picture-raised $\tilde{b}_B$ ghost and satisfy

$$\begin{aligned} \{Q, G^\alpha\} = \lambda^\alpha T, \quad [Q, H^{[\alpha\beta]}] = \lambda^{[\alpha} G^{\beta]}, \quad \{Q, K^{[\alpha\beta\gamma]}\} = \lambda^{[\alpha} H^{\beta\gamma]}\,, \\ [Q, L^{[\alpha\beta\gamma\delta]}] = \lambda^{[\alpha} K^{\beta\gamma\delta]}\,, \qquad \lambda^{[\alpha} L^{\beta\gamma\delta\kappa]} = 0\,. \end{aligned} \tag{3.13}$$

In addition to satisfying $\{Q_{\rm nonmin}, b_{\rm nonmin}\} = T_{\rm nonmin}$, one can verify that $b_{\rm nonmin}$ has no poles with itself. Note that only the antisymmetrized components of $H^{\alpha\beta}$, $K^{\alpha\beta\gamma}$ and $L^{\alpha\beta\gamma\delta}$ contribute to $b_{\rm nonmin}$, which makes the computation of coefficients in $b_{\rm nonmin}$ much simpler than in the computation of the picture-raised $\tilde{b}_B$ ghost [4, 39, 28]. Although $b_{\rm nonmin}$ appears complicated in (3.11), its construction in terms of Siegel-like constraints [32] suggests that it may have a natural superspace interpretation.

To complete the construction of the $\hat{c} = 3$ $N = 2$ generators, one needs to construct the U(1) current $J_{\rm nonmin}$ by computing the double pole of $b_{\rm nonmin}$ with the integrand of $Q_{\rm nonmin}$. The result is

$$J_{\rm nonmin} = w_\alpha \lambda^\alpha - s^\alpha r_\alpha - 2\frac{\overline{\lambda}_\alpha \partial\lambda^\alpha + r_\alpha \partial\theta^\alpha}{(\overline{\lambda}\lambda)} + 2\frac{(\overline{\lambda}_\alpha r^\alpha)(\overline{\lambda}_\beta \partial\theta^\beta)}{(\overline{\lambda}\lambda)^2}\,. \tag{3.14}$$

The unusual non-quadratic terms in $J_{\rm nonmin}$ can be understood to be necessary for two reasons. Firstly, the term $(\overline{\lambda}_\alpha G^\alpha)/(\overline{\lambda}\lambda)$ in $b_{\rm nonmin}$ has a double pole with $\lambda^\alpha w_\alpha$, which needs to be cancelled by the double pole of $b_{\rm nonmin}$ with the non-quadratic terms in order that $b_{\rm nonmin}$ is a U(1) primary field. Secondly, the triple pole of $J_{\rm nonmin}$ with $T_{\rm nonmin}$ of (3.12) is equal to $-8 + 11 = +3$. But the $N = 2$ Jacobi identities imply that this ghost-number anomaly of $+3$ should be equal to the double pole of $J_{\rm nonmin}$ with itself, which gives the value $-4 + 11 = +7$ if one does not include the contribution from the non-quadratic terms.

So the twisted $\hat{c} = 3$ $N = 2$ generators are given by the U(1) current $J_{\rm nonmin}$ of (3.14), the fermionic generators $\lambda^\alpha d_\alpha + \overline{w}^\alpha r_\alpha$ and $b_{\rm nonmin}$ of (3.11), and the stress tensor $T_{\rm nonmin}$ of (3.12). Although the form of $J_{\rm nonmin}$ is complicated, it can be simplified by shifting by

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a BRST-trivial quantity as

$$\begin{aligned} J'_{\text{nonmin}} &= J_{\text{nonmin}} + \left\{ Q_{\text{nonmin}}, -s^\alpha \overline{\lambda}_\alpha + 2\frac{\overline{\lambda}_\alpha \partial\theta^\alpha}{(\overline{\lambda}\lambda)} \right\} \\ &= J_{\text{nonmin}} - \overline{w}^\alpha \overline{\lambda}_\alpha + s^\alpha r_\alpha + 2\frac{\overline{\lambda}_\alpha \partial\lambda^\alpha + r_\alpha \partial\theta^\alpha}{(\overline{\lambda}\lambda)} - 2\frac{(\overline{\lambda}_\alpha r^\alpha)(\overline{\lambda}_\beta \partial\theta^\beta)}{(\overline{\lambda}\lambda)^2} \\ &= w_\alpha \lambda^\alpha - \overline{w}^\alpha \overline{\lambda}_\alpha \,. \end{aligned} \tag{3.15}$$

Although J'_{nonmin} has double poles with b_{nonmin} and does not have level +3, one can easily check that $[\int dz J'_{\text{nonmin}}, Q_{\text{nonmin}}] = Q_{\text{nonmin}}$ and $[\int dz J'_{\text{nonmin}}, b_{\text{nonmin}}] = -b_{\text{nonmin}}$. Furthermore, it will be shown in the appendix that the triple pole of J'_{nonmin} with T_{nonmin} is +3, so the ghost-number anomaly is preserved using J'_{nonmin}. These are the only necessary conditions for the ghost current in critical topological string theory, as can be seen by comparing with the ghost current of the bosonic string, $J = bc$, which has double poles with the BRST current and whose level of +1 does not coincide with its ghost-number anomaly of +3. So there is no problem with replacing J_{nonmin} by J'_{nonmin} in the definition of the topological string associated to the non-minimal pure spinor formalism.

In the next section, superstring scattering amplitudes will be computed using topological methods with the U(1) charge $\int dz J = \int dz (w_\alpha \lambda^\alpha - \overline{w}^\alpha \overline{\lambda}_\alpha)$, the BRST operator $Q = \int dz (\lambda^\alpha d_\alpha + \overline{w}^\alpha r_\alpha)$, the stress tensor $T = -\frac{1}{2}\partial x^m \partial x_m - p_\alpha \partial\theta^\alpha + w_\alpha \partial\lambda^\alpha + \overline{w}^\alpha \partial\overline{\lambda}_\alpha - s^\alpha \partial r_\alpha$, and the b ghost

$$\begin{aligned} b &= s^\alpha \partial\overline{\lambda}_\alpha + \frac{\overline{\lambda}_\alpha (2\Pi^m (\gamma_m d)^\alpha - N_{mn}(\gamma^{mn}\partial\theta)^\alpha - J_\lambda \partial\theta^\alpha - \partial^2\theta^\alpha)}{4(\overline{\lambda}\lambda)} + \\ &+ \frac{(\overline{\lambda}\gamma^{mnp} r)(d\gamma_{mnp} d + 24 N_{mn}\Pi_p)}{192(\overline{\lambda}\lambda)^2} - \frac{(r\gamma_{mnp} r)(\overline{\lambda}\gamma^m d) N^{np}}{16(\overline{\lambda}\lambda)^3} + \\ &+ \frac{(r\gamma_{mnp} r)(\overline{\lambda}\gamma^{pqr} r) N^{mn} N_{qr}}{128(\overline{\lambda}\lambda)^4} \,. \end{aligned} \tag{3.16}$$

Note that for the rest of this paper, the subscript nonmin will be dropped from these operators.

4. Computation of scattering amplitudes

4.1 Tree amplitudes

Since the non-minimal pure spinor formalism is a $\hat{c} = 3$ $N = 2$ string theory, one can use standard methods developed for critical topological strings to compute scattering amplitudes. For example, N-point tree amplitudes are computed as in bosonic string theory by the correlation function of three unintegrated vertex operators V satisfying $QV = 0$ and $N - 3$ integrated vertex operators $\int dz\, \mathrm{U}(z)$ satisfying $QU = \partial V$. As in the minimal pure spinor formalism, functional integration over the worldsheet variables of +1 conformal weight is straightforward using the poles in the OPE's of (2.6) and (3.7). One is then left with an expression $\mathcal{A} = \langle f(\lambda, \overline{\lambda}, r, \theta) \rangle$ where $f(\lambda, \overline{\lambda}, r, \theta)$ carries +3 U(1) charge and depends only on the zero modes of λ^α, $\overline{\lambda}_\alpha$, r_α and θ^α. Note that integration over the x^m zero modes is performed in the standard manner and will be ignored throughout this paper.

Since λ^α and $\overline{\lambda}_\alpha$ are non-compact bosonic variables, the integral over the zero modes

$$\mathcal{A} = \int [d\lambda][d\overline{\lambda}][dr]d^{16}\theta f(\lambda,\overline{\lambda},r,\theta) \tag{4.1}$$

needs to be regularized. A useful regularization method developed by Marnelius [40] for BRST-invariant systems involves inserting the factor $\mathcal{N} = \exp(\{Q,\chi\})$ into the integral where χ is some fermionic function of the worldsheet variables. Since $f(\lambda,\overline{\lambda},r,\theta)$ is BRST-invariant and $\mathcal{N} = 1 + \cdots$ where $\cdots$ is BRST-trivial, the integral will be independent of the choice of χ.

In the non-minimal pure spinor formalism, it is convenient to choose $\chi = -\overline{\lambda}_\alpha\theta^\alpha$ so that

$$\mathcal{N} = \exp(\{Q,\chi\}) = \exp(-\overline{\lambda}_\alpha\lambda^\alpha - r_\alpha\theta^\alpha)\,. \tag{4.2}$$

Treating $\overline{\lambda}_\alpha$ as the complex conjugate of λ^α, the expression

$$\mathcal{A} = \int [d\lambda][d\overline{\lambda}][dr]d^{16}\theta\ \mathcal{N} f(\lambda,\overline{\lambda},r,\theta) \tag{4.3}$$

is well-defined if one assumes that $f(\lambda,\overline{\lambda},r,\theta)$ does not diverge too fast as $\lambda\overline{\lambda}\to 0$.

To determine how fast $f(\lambda,\overline{\lambda},r,\theta)$ is allowed to diverge as $\lambda\overline{\lambda}\to 0$, note that the measure factors $[d\lambda]$ and $[d\overline{\lambda}]$ for pure spinors satisfy[4, 41]

$$[d\lambda]\lambda^\beta\lambda^\gamma\lambda^\delta = (\epsilon T^{-1})^{\beta\gamma\delta}_{\alpha_1\ldots\alpha_{11}} d\lambda^{\alpha_1}\ldots d\lambda^{\alpha_{11}} \tag{4.4}$$

and

$$[d\overline{\lambda}]\overline{\lambda}_\beta\overline{\lambda}_\gamma\overline{\lambda}_\delta = (\epsilon T)^{\alpha_1\ldots\alpha_{11}}_{\beta\gamma\delta} d\overline{\lambda}_{\alpha_1}\ldots d\overline{\lambda}_{\alpha_{11}}\,,$$

where $(\epsilon T^{-1})^{\beta\gamma\delta}_{\alpha_1\ldots\alpha_{11}}$ and $(\epsilon T)^{\alpha_1\ldots\alpha_{11}}_{\beta\gamma\delta}$ are Lorentz-invariant tensors defined in [1, 4] which are antisymmetric in $[\alpha_1\ldots\alpha_{11}]$ and are symmetric and gamma-matrix traceless in $(\beta\gamma\delta)$. Up to an overall normalization constant,

$$(\epsilon T)^{\alpha_1\ldots\alpha_{11}}_{\beta\gamma\delta} = \epsilon^{\alpha_1\ldots\alpha_{16}}\gamma^m_{\alpha_{12}\rho}\gamma^n_{\alpha_{13}\sigma}\gamma^p_{\alpha_{14}\tau}\left((\gamma_{mnp})_{\alpha_{15}\alpha_{16}}(\delta^\rho_{(\beta}\delta^\sigma_\gamma\delta^\tau_{\delta)} - \frac{1}{40}\gamma^m_{(\beta\gamma}\delta^\rho_{\delta)}\gamma^{\sigma\tau}_m\right)\,. \tag{4.5}$$

Furthermore, the constraint $\overline{\lambda}\gamma^m r = 0$ implies that the measure factor $[dr]$ satisfies

$$[dr] = (\epsilon T^{-1})^{\beta\gamma\delta}_{\alpha_1\ldots\alpha_{11}}\,\overline{\lambda}_\beta\overline{\lambda}_\gamma\overline{\lambda}_\delta\left(\frac{\partial}{\partial r_{\alpha_1}}\right)\cdots\left(\frac{\partial}{\partial r_{\alpha_{11}}}\right)\,. \tag{4.6}$$

So the measure factor $[d\lambda][d\overline{\lambda}][dr]$ goes like $\lambda^8\overline{\lambda}^{11}$ as $\lambda\overline{\lambda}\to 0$, which implies that $f(\lambda,\overline{\lambda},r,\theta)$ must diverge slower than $\lambda^{-8}\overline{\lambda}^{-11}$ in order that (4.3) is well-defined. If one wants to compute amplitudes in which $f(\lambda,\overline{\lambda},r,\theta)$ diverges as fast as $\lambda^{-8}\overline{\lambda}^{-11}$ when $\lambda\overline{\lambda}\to 0$, an alternative regularization method for the zero modes must be found.

The restriction that $f(\lambda,\overline{\lambda},r,\theta)$ diverges slower than $\lambda^{-8}\overline{\lambda}^{-11}$ is related to the operator $\xi = (\overline{\lambda}\theta)/(\overline{\lambda}\lambda + r\theta)$ which satisfies $Q\xi = 1$. Since $QV = 0$ implies that $Q(\xi V) = V$, the existence of the operator ξ naively implies that the BRST cohomology is trivial. For example, $f = Q(\xi f)$ where $f = (\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta)$ naively implies that

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$\langle \mathcal{N} f\rangle = \langle \mathcal{N} Q(\xi f)\rangle = 0$. But because of the restriction that f diverges slower than $\lambda^{-8}\overline{\lambda}^{-11}$, $\langle \mathcal{N} Q(\Omega)\rangle$ is only guaranteed to vanish if Ω diverges slower than $\lambda^{-8}\overline{\lambda}^{-10}$ as $\lambda\overline{\lambda} \to 0$. When $f = (\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\lambda\gamma^p\theta)(\theta\gamma_{mnp}\theta)$, ξf contains terms which diverge as $\lambda^{-8}\overline{\lambda}^{-10}(\theta)^{16}(r)^{10}$. So ξf is not an allowable gauge parameter, which explains why $\langle \mathcal{N} f\rangle \neq 0$.

So the regularized prescription for computing the N-point tree amplitude using topological string methods is given by the correlation function

$$\mathcal{A} = \left\langle \mathcal{N}(y) V_1(z_1) V_2(z_2) V_3(z_3) \int dz_4 U_4(z_4) \dots \int dz_N U_N(z_N) \right\rangle, \tag{4.7}$$

where $\mathcal{N}(y) = \exp(\{Q, \chi(y)\}) = \exp(-\lambda(y)\overline{\lambda}(y) - r(y)\theta(y))$ and y is an arbitrary point on the worldsheet. Suppose that all external states are chosen in the gauge where the vertex operators V and U are independent of the non-minimal fields. Then after integrating out the variables of $+1$ conformal weight using the poles in their OPE's, one obtains

$$\mathcal{A} = \langle \mathcal{N} f(\lambda, \theta)\rangle = \langle \mathcal{N} \lambda^\alpha \lambda^\beta \lambda^\gamma f_{\alpha\beta\gamma}(\theta)\rangle\,, \tag{4.8}$$

which has no divergences when $\lambda\overline{\lambda} \to 0$. Using the measure factors defined above, one finds up to an overall normalization constant that

$$\begin{aligned}
\mathcal{A} &= \int [d\lambda][d\overline{\lambda}][dr] d^{16}\theta \exp(-\overline{\lambda}_\alpha \lambda^\alpha - r_\alpha \theta^\alpha) \lambda^\beta \lambda^\gamma \lambda^\delta f_{\beta\gamma\delta}(\theta) \\
&= \int d^{16}\theta (\epsilon T^{-1})^{\beta\gamma\delta}_{\alpha_1 \dots \alpha_{11}} \theta^{\alpha_1} \dots \theta^{\alpha_{11}} f_{\beta\gamma\delta}(\theta) \\
&= \epsilon^{\alpha_1 \dots \alpha_{16}} \left(\epsilon T^{-1}\right)^{\beta\gamma\delta}_{\alpha_1 \dots \alpha_{11}} \left(\frac{\partial}{\partial\theta^{\alpha_{12}}}\right) \dots \left(\frac{\partial}{\partial\theta^{\alpha_{16}}}\right) f_{\beta\gamma\delta}(\theta)\,,
\end{aligned} \tag{4.9}$$

which agrees with the result from the minimal pure spinor formalism.

To understand the relationship between the non-minimal and minimal computations, note that BRST-invariance implies that the amplitude is unaffected by rescaling $\chi = -\overline{\lambda}_\alpha \theta^\alpha$ to $\chi = -\rho\overline{\lambda}_\alpha \theta^\alpha$ for any positive ρ in the definition of $\mathcal{N}$. So one can take the limit $\rho \to \infty$ in $\mathcal{N}_\rho(y) = \exp(-\rho(\lambda(y)\overline{\lambda}(y) + r(y)\theta(y)))$, which is non-vanishing only when $\lambda^\alpha(y) = \overline{\lambda}_\alpha(y) = 0$. So in the limit $\rho \to \infty$, $\mathcal{N}_\rho(y)$ contains the same $\delta^{11}(\lambda)$ dependence as the product of eleven picture-lowering operators $\prod_{I=1}^{11} Y_{C_I}(y)$ in the minimal formalism. However, in addition to being manifestly Lorentz-invariant, the advantage of using $\mathcal{N}(y)$ instead of picture-changing operators is that one can take the opposite limit $\rho \to 0$ in which $\mathcal{N}_\rho(y)$ becomes a smooth invertible function.

After introducing the regularization factor $\mathcal{N} = \exp(-\lambda\overline{\lambda} - r\theta)$, one can also define N-point tree amplitudes in a worldsheet reparameterization invariant manner as

$$\mathcal{A} = \left\langle \mathcal{N}(y)\; V_1(z_1) \dots V_N(z_N) \int dz_4 b(z_4) \dots \int dz_N b(z_N) \right\rangle, \tag{4.10}$$

where $b(z)$ is defined in (3.16). But since each unintegrated vertex operator V goes like λ and each b ghost goes like $\overline{\lambda}/(\overline{\lambda}\lambda)^4$, $f(\lambda, \overline{\lambda}, r, \theta)$ goes like $\lambda^3(\overline{\lambda}\lambda)^{9-3N}$ when $\lambda\overline{\lambda} \to 0$. Since $f(\lambda, \overline{\lambda}, r, \theta)$ must diverge slower than $\lambda^{-8}\overline{\lambda}^{-11}$, a maximum of three b ghosts (or six unintegrated vertex operators) can be allowed in computations using this regularization method.

4.2 Loop amplitudes

To compute N-point g-loop amplitudes, one uses the topological prescription

$$\mathcal{A} = \int d^{3g-3}\tau \left\langle \mathcal{N}(y) \prod_{j=1}^{3g-3} \left(\int dw_j \mu_j(w_j) b(w_j) \right) \prod_{r=1}^{N} \int dz_r \, \mathrm{U}(z_r) \right\rangle , \tag{4.11}$$

where τ_j are the complex Teichmuller parameters and μ_j are the associated Beltrami differentials, $b(z)$ is defined in (3.16), and $\mathcal{N}(y)$ is a regularization factor for the genus g zero modes which will be defined below. To define this regularization factor, first separate off the zero modes of the gauge-invariant worldsheet fields of +1 conformal weight as

$$\begin{aligned}
N_{mn}(z) &= \widehat{N}_{mn}(z) + \sum_{I=1}^{g} N^I_{mn}\omega_I(z) , \qquad \overline{N}_{mn}(z) = \widehat{\overline{N}}_{mn}(z) + \sum_{I=1}^{g} \overline{N}^I_{mn}\omega_I(z) , \\
J_\lambda(z) &= \widehat{J}_\lambda(z) + \sum_{I=1}^{g} J^I_\lambda \omega_I(z) , \qquad \overline{J}_{\overline{\lambda}}(z) = \widehat{\overline{J}}_{\overline{\lambda}}(z) + \sum_{I=1}^{g} \overline{J}^I_{\overline{\lambda}} \omega_I(z) , \\
d_\alpha(z) &= \widehat{d}_\alpha(z) + \sum_{I=1}^{g} d^I_\alpha \omega_I(z) , \\
S_{mn}(z) &= \widehat{S}_{mn}(z) + \sum_{I=1}^{g} S^I_{mn}\omega_I(z) , \qquad S(z) = \widehat{S}(z) + \sum_{I=1}^{g} S^I \omega_I(z) ,
\end{aligned} \tag{4.12}$$

where $\omega_I(z)$ are the g holomorphic one-forms satisfying $\int_{A_I} dz \; \omega_J(z) = \delta_{IJ}$, $\int_{A_I} dz$ are contour integrals around the g non-trivial A-cycles, and the hatted variables $\widehat{F}(z)$ of (4.12) have no zero modes and are defined to satisfy $\int_{A_I} dz \widehat{F}(z) = 0$ for $I = 1$ to g.

As in multiloop calculations using the minimal pure spinor formalism [4], one can use the poles in the OPE's of (2.6) and (3.7) for the hatted variables to perform the functional integral over the non-zero modes. Note that the partition function for the non-zero modes is equal to one since there are an equal number of bosons and fermions at +1 conformal weight.

After integrating out the non-zero modes, one obtains

$$\mathcal{A} = \left\langle \mathcal{N} f(\lambda, \overline{\lambda}, r, \theta, N^I_{mn}, \overline{N}^I_{mn}, J^I_\lambda, \overline{J}^I_{\overline{\lambda}}, d^I_\alpha, S^I_{mn}, S^I) \right\rangle , \tag{4.13}$$

where f is some BRST-invariant function of the zero modes with U(1) charge $3 - 3g$. To regularize this integral over the zero modes, the factor $\mathcal{N}(y)$ will be chosen as $\mathcal{N}(y) = \exp(\{Q, \chi(y)\})$ where

$$\chi(y) = -\overline{\lambda}_\alpha(y)\theta^\alpha(y) - \sum_{I=1}^{g} \left(\frac{1}{2} N^I_{mn} S^{mnI} + J^I_\lambda S^I \right) . \tag{4.14}$$

Using the BRST transformations

$$\begin{aligned}
\{Q, S^I_{mn}\} &= \overline{N}^I_{mn} , \qquad \{Q, S^I\} = \overline{J}^I_{\overline{\lambda}} , \\
[Q, N^I_{mn}] &= -\frac{1}{2} \int_{A_I} dz \lambda \gamma_{mn} d , \qquad [Q, J^I_\lambda] = \int_{A_I} dz \lambda^\alpha d_\alpha ,
\end{aligned} \tag{4.15}$$

one finds that

$$\mathcal{N}(y) = \exp(-\overline{\lambda}_\alpha(y)\lambda^\alpha(y) - r_\alpha(y)\theta^\alpha(y)) \times$$
$$\times \exp\left(\sum_{I=1}^{g}\left[-\frac{1}{2}N^I_{mn}\overline{N}^{mnI} - J^I_\lambda \overline{J}^I_{\overline{\lambda}} - \frac{1}{4}S^I_{mn}\int_{A_I} dz\ \lambda\gamma^{mn}d + S^I\int_{A_I} dz\lambda^\alpha d_\alpha\right]\right). \tag{4.16}$$

So one needs to compute the integral over the zero modes

$$\mathcal{A} = \int [d\lambda][d\overline{\lambda}][dr]d^{16}\theta \prod_{I=1}^{g}[dw^I][d\overline{w}^I][ds^I]d^{16}d^I\ \ \mathcal{N}f\,. \tag{4.17}$$

Using the methods of [4], one can show that the measure factors $[dw^I]$, $[d\overline{w}^I]$, and $[ds^I]$ are defined as

$$\begin{aligned}
[dw^I]\lambda^{\alpha_1}\dots\lambda^{\alpha_8} &= M^{\alpha_1\dots\alpha_8}_{m_1n_1\dots m_{10}n_{10}} dN^{m_1n_1I}\dots dN^{m_{10}n_{10}I}dJ^I_\lambda\,,\\
[d\overline{w}^I]\overline{\lambda}_{\alpha_1}\dots\overline{\lambda}_{\alpha_8} &= (M^{-1})^{m_1n_1\dots m_{10}n_{10}}_{\alpha_1\dots\alpha_8} d\overline{N}^I_{m_1n_1}\dots d\overline{N}^I_{m_{10}n_{10}}d\overline{J}^I_{\overline{\lambda}}\,,\\
[ds^I] &= M^{\alpha_1\dots\alpha_8}_{m_1n_1\dots m_{10}n_{10}}\overline{\lambda}_{\alpha_1}\dots\overline{\lambda}_{\alpha_8}\frac{\partial}{\partial S^I_{m_1n_1}}\cdots\frac{\partial}{\partial S^I_{m_{10}n_{10}}}\frac{\partial}{\partial S^I}\,,
\end{aligned} \tag{4.18}$$

where $M^{\alpha_1\dots\alpha_8}_{m_1n_1\dots m_{10}n_{10}}$ is a Lorentz-invariant tensor which is antisymmetric after switching m_j with n_j, antisymmetric after switching $[m_jn_j]$ with $[m_kn_k]$, and symmetric and gamma-matrix traceless in $(\alpha_1\dots\alpha_8)$. Up to an overall normalization constant,

$$\begin{aligned}
M^{\alpha_1\dots\alpha_8}_{m_1n_1\dots m_{10}n_{10}}\overline{\lambda}_{\alpha_1}\dots\overline{\lambda}_{\alpha_8}\psi^{m_1n_1}\dots\psi^{m_{10}n_{10}} &= (\overline{\lambda}\gamma_{m_1n_1m_2m_3m_4}\overline{\lambda})(\overline{\lambda}\gamma_{m_5n_5n_2m_6m_7}\overline{\lambda})\times\\
&\times(\overline{\lambda}\gamma_{m_8n_8n_3n_6m_9}\overline{\lambda})(\overline{\lambda}\gamma_{m_{10}n_{10}n_4n_7n_9}\overline{\lambda})\times\\
&\times\psi^{m_1n_1}\dots\psi^{m_{10}n_{10}}\,,
\end{aligned} \tag{4.19}$$

where $\psi_{m_jn_j}$ are fermionic antisymmetric two-forms.

As long as f does not diverge too fast as $\lambda\overline{\lambda}\to 0$, the regularized expression of (4.17) is well-defined. For example, if f is assumed to be independent of S^I_{mn} and S^I, then all $11g$ zero modes for these fermionic variables must come from the regularization factor $\mathcal{N}$ of (4.16). Each of these zero modes is multiplied by a factor of $(\lambda\gamma_{mn}d)$ or $(\lambda^\alpha d_\alpha)$, so $\mathcal{N}$ contributes a factor which goes like λ^{11g} as $\lambda\overline{\lambda}\to 0$. Since $[d\lambda][d\overline{\lambda}][dr]\to\lambda^8\overline{\lambda}^{11}$ and $\prod_{I=1}^g[dw^I][d\overline{w}^I][ds^I]\to\lambda^{-8g}$, $\int[d\lambda][d\overline{\lambda}][dr]\prod_{I=1}^g[dw^I][d\overline{w}^I][ds^I]\mathcal{N}$ goes like $\lambda^{8+3g}\overline{\lambda}^{11}$ as $\lambda\overline{\lambda}\to 0$.

So f must diverge slower than $\lambda^{-8-3g}\overline{\lambda}^{-11}$ as $\lambda\overline{\lambda}\to 0$ in order that (4.17) is well-defined. Since each b ghost goes like $\overline{\lambda}/(\overline{\lambda}\lambda)^4$ as $\lambda\overline{\lambda}\to 0$, the regularization method described here is valid for three or fewer b ghosts, i.e. for amplitudes up to two loops. To compute amplitudes with more than two loops using the topological string methods described here, one needs to find an alternative regularization method for the zero modes. Work is currently in progress with Nikita Nekrasov on finding such a method.

To check the consistency of this computational method, consider the zero mode structure of four-point massless one-loop and two-loop amplitudes. At one-loop, there is one b ghost of (3.16), one unintegrated vertex operator $V=\lambda^\alpha A_\alpha(x,\theta)$, and three integrated vertex operators

$$U = \int dz(\partial\theta^\alpha A_\alpha + \Pi^m B_m + d_\alpha W^\alpha + N_{mn}F^{mn}), \tag{4.20}$$

where (A_α, B_m) are the spinor and vector gauge superfields and (W^α, F_{mn}) are the spinor and vector field-strengths of super-Yang-Mills. To absorb the 16 d_α and 11 s^α fermionic zero modes, $\mathcal{N}$ must contribute 11 d_α and 11 s^α zero modes, the b ghost must contribute 2 d_α zero modes through the term $(\overline{\lambda}\gamma^{mnp}r)(d\gamma_{mnp}d)/(\overline{\lambda}\lambda)^2$, and each of the three integrated vertex operators must contribute a d_α zero mode through the term $\int dz d_\alpha W^\alpha$. After integrating over the r_α zero modes, the amplitude is proportional to

$$\int d^{16}\theta(\theta)^{10}AWWW, \tag{4.21}$$

where the Lorentz contractions of the spinor indices has not yet been worked out. However, by dimensional analysis, one see that (4.21) has the correct zero mode structure to contribute an F^4 term for open strings, or an R^4 term for closed strings after taking the holomorphic square.

For four-point two-loop massless amplitudes, there are three b ghosts of (3.16) and four integrated vertex operators of (4.20). To absorb the 32 d_α and 22 s^α fermionic zero modes, $\mathcal{N}$ must contribute 22 d_α and 22 s^α zero modes, each of the three b ghosts must contribute 2 d_α zero modes through the term $(\overline{\lambda}\gamma^{mnp}r)(d\gamma_{mnp}d)/(\overline{\lambda}\lambda)^2$, and each of the four integrated vertex operators must contribute a d_α zero mode through the term $\int dz d_\alpha W^\alpha$. After integrating over the r_α zero modes, the amplitude is proportional to

$$\int d^{16}\theta(\theta)^{8}WWWW, \tag{4.22}$$

which has the correct zero mode structure to contribute a $\partial^2 F^4$ term for open strings, or a $\partial^4 R^4$ term for closed strings after taking the holomorphic square. It should not be too difficult to verify if the contractions of the Lorentz indices in (4.21) and (4.22) reproduce the appropriate t_8 index contractions in the R^4 and $\partial^4 R^4$ terms.

5. Cubic open superstring field theory

Using the RNS formalism for the superstring, cubic open superstring field theory actions require midpoint insertions which cause contact-term divergences or gauge invariance problems. For example, in the cubic Neveu-Schwarz action of [42],

$$S = \left\langle \frac{1}{2}VQV + \frac{1}{3}VVVZ\left(\frac{\pi}{2}\right)\right\rangle, \tag{5.1}$$

where the open string fields V are multiplied using Witten's star product, V is chosen in the -1 picture, and $Z(\frac{\pi}{2})$ is the picture-raising operator inserted at the string midpoint. Since $Z(y)Z(z)$ is divergent when $y \to z$, the action produces unphysical contact-term divergences when interaction points collide [43, 44]. Alternatively, in the cubic Neveu-

Schwarz action of [45, 46],

$$S = \left\langle \left(\frac{1}{2} VQV + \frac{1}{3} VVV \right) Y^2 \left(\frac{\pi}{2} \right) \right\rangle, \tag{5.2}$$

where V is chosen in the zero picture and $Y^2(\frac{\pi}{2})$ is the square of the picture-lowering operator inserted at the string midpoint. Although the action of (5.2) does not have contact-term divergences, it has gauge invariance problems since the linearized equation of motion is $Y^2(\frac{\pi}{2})QV = 0$ instead of $QV = 0$. Since $Y^2(\frac{\pi}{2})$ has a non-trivial kernel, the equation $Y^2(\frac{\pi}{2})QV = 0$ has additional solutions given by $V = Ker(Y^2(\frac{\pi}{2}))$. If one projects out states in the kernel of $Y^2(\frac{\pi}{2})$ to remove these unwanted solutions from the Hilbert space, the associativity property of the star-product is ruined and gauge invariance is broken [47, 48].

Although these problems are avoided in the non-polynomial WZW-like action for open superstring field theory [49] which does not require midpoint insertions, it would be useful to have a cubic open superstring field theory action. Since the equation of motion in the pure spinor formalism for the open superstring field V is

$$QV + VV = 0, \tag{5.3}$$

a natural suggestion [8] is to use the Chern-Simons-like action

$$S = \left\langle \frac{1}{2} VQV + \frac{1}{3} VVV \right\rangle \tag{5.4}$$

of bosonic string field theory. However, using the minimal pure spinor formalism of [1], the inner product for zero modes defined by

$$\langle 0 | (\lambda\gamma_m\theta)(\lambda\gamma_n\theta)(\lambda\gamma_p\theta)(\theta\gamma^{mnp}\theta) | 0 \rangle = 1 \tag{5.5}$$

is degenerate, so the action of (5.4) does not generate the equations of (5.3). Since the norm is degenerate, $\langle A|B\rangle = 0$ for every string field $|B\rangle$ does not imply that $|A\rangle = 0$. For example, $|A\rangle = (\theta)^n|0\rangle$ for $n > 5$ satisfies $\langle A|B\rangle = 0$ for any string field $|B\rangle$. Therefore, using the minimal inner product of (5.5), the action of (5.4) does not imply that components of $(QV + VV)$ with more than five θ's must vanish on-shell.

As shown in [4], the inner product for zero modes in the minimal pure spinor formalism can be made non-degenerate by defining

$$\langle 0 | f(\lambda, \theta) | 0 \rangle = \int [d\lambda] d^{16}\theta f(\lambda, \theta), \tag{5.6}$$

where $[d\lambda]$ is defined in (4.4). This implies that

$$\langle 0 | (\lambda\gamma_m\theta)(\lambda\gamma_n\theta)(\lambda\gamma_p\theta)(\theta\gamma^{mnp}\theta) \prod_{I=1}^{11} Y_{C_I} | 0 \rangle$$

is non-zero where $Y_{C_I} = (C^I_\alpha \theta^\alpha)\delta(C^I_\beta \lambda^\beta)$ is the picture-lowering operator and C^I_α are constant spinors for $I = 1$ to 11. Using this non-degenerate norm, the appropriate open superstring field theory action would be

$$S = \left\langle \left(\frac{1}{2}VQV + \frac{1}{3}VVV\right) \prod_{I=1}^{11} Y_{C_I}(\frac{\pi}{2}) \right\rangle, \tag{5.7}$$

where the eleven picture-lowering operators are inserted at the string midpoint. However, in addition to causing gauge-invariance problems as in the RNS cubic action of (5.2), these midpoint insertions break Lorentz invariance because of their explicit dependence on C^I_α.

As discussed in the previous section, the non-minimal pure spinor formalism does not require picture-changing operators but instead introduces the regularization factor $\mathcal{N} = \exp(-\lambda\overline{\lambda} - r\theta)$. Since the inner product for zero modes defined by

$$\langle 0|\mathcal{N} f(\lambda, \overline{\lambda}, r, \theta)|0\rangle = \int [d\lambda][d\overline{\lambda}][dr]d^{16}\theta f(\lambda, \overline{\lambda}, r, \theta)\exp(-\overline{\lambda}_\alpha\lambda^\alpha - r_\alpha\theta^\alpha) \tag{5.8}$$

is non-degenerate, the cubic action

$$S = \left\langle \left(\frac{1}{2}VQV + \frac{1}{3}VVV\right) \mathcal{N}\left(\frac{\pi}{2}\right) \right\rangle \tag{5.9}$$

generates the equation of motion

$$\mathcal{N}\left(\frac{\pi}{2}\right)(QV + VV) = 0\,, \tag{5.10}$$

where the regularization factor $\mathcal{N}(y)$ is inserted at the string midpoint. But unlike the picture-lowering operator in (5.2) or (5.7), $\mathcal{N}$ has no kernel since $\mathcal{N}^{-1} = \exp(\lambda\overline{\lambda} + r\theta)$ is well-defined even when acting on off-shell states. So there are no gauge invariance problems and (5.10) implies the desired equation of motion $QV + VV = 0$.

Note that the action of (5.9) is manifestly Lorentz invariant, but is not manifestly spacetime supersymmmetric because of the explicit θ dependence in the regularization factor $\mathcal{N} = \exp(-\overline{\lambda}_\alpha\lambda^\alpha - r_\alpha\theta^\alpha)$. The action differs from the "minimal" cubic action of (5.4) since the string field V can depend on the non-minimal variables $\overline{\lambda}_\alpha$ and r_α. Although the linearized on-shell string field is independent of these non-minimal variables, the off-shell dependence on the non-minimal variables is necessary for generating the $(\theta)^n$ components for $n > 5$ of the equation of motion $QV + VV = 0$.

Although the discussion of the inner product has focused up to now on the zero mode dependence of the string field V, it is easy to see that the non-zero modes do not cause any problems. To evaluate the cubic action of (5.9) for an arbitrary string field V, first convert the string field to a vertex operator on the disk, and then use the conformal field theory OPE's of (2.6) and (3.7) for the variables of +1 conformal weight to functionally integrate over the non-zero modes. The remaining dependence on the zero modes is integrated using the regularization factor $\mathcal{N} = \exp(-\overline{\lambda}_\alpha\lambda^\alpha - r_\alpha\theta^\alpha)$ as in (5.8). Since the string field V will be required to be non-singular as $\lambda\overline{\lambda} \to 0$, the integral $\int[d\lambda][d\overline{\lambda}][dr]d^{16}\theta\; \mathcal{N} f(\lambda, \overline{\lambda}, r, \theta)$ is guaranteed to be well-defined.

For BRST-invariant external states, rescaling the regularization factor as

$$\mathcal{N} = \exp(-\overline{\lambda}\lambda - r\theta) \to \mathcal{N}_\rho = \exp(-\rho(\overline{\lambda}\lambda + r\theta))$$

for any positive ρ does not affect the scattering amplitude. However, since the string field V is off-shell, the cubic open superstring field theory action will depend on the scaling factor ρ. To make this dependence explicit, define the BRST-invariant charge

$$j_{\overline{\lambda}} = \int dz \overline{J}_{\overline{\lambda}} = \int dz (\overline{w}^\alpha \overline{\lambda}_\alpha - s^\alpha r_\alpha) \tag{5.11}$$

such that $\overline{\lambda}_\alpha$ and r_α carry $+1$ charge and $\overline{w}_\alpha$ and s^α carry -1 charge. Since $\overline{w}^\alpha$ and s^α can only appear in the $j_{\overline{\lambda}}$-neutral combinations of (3.4) and (3.6), all states in the Hilbert space carry non-negative $j_{\overline{\lambda}}$ charge. And since $[Q, j_{\overline{\lambda}}] = 0$, the cubic action of (5.9) can be written as $S(\rho) = \sum_{m=0}^{\infty} S_m(\rho)$ where

$$S_m(\rho) = \left\langle \left(\frac{1}{2} \sum_{p=0}^{m} V_p Q V_{m-p} + \frac{1}{3} \sum_{p=0}^{m} \sum_{q=0}^{m-p} V_p V_q V_{m-p-q} \right) \mathcal{N}_\rho \left(\frac{\pi}{2} \right) \right\rangle \tag{5.12}$$

and V_q is a string field satisfying $j_{\overline{\lambda}}(V_q) = q V_q$. Under the scaling of $\overline{\lambda}_\alpha \to c\overline{\lambda}_\alpha$ and $r_\alpha \to c r_\alpha$, one can easily verify that $\mathcal{N}_\rho \to \mathcal{N}_{c\rho}$, $V_q \to c^q V_q$, and the measure factor $[d\overline{\lambda}][dr]$ is invariant. This implies that $S_m(\rho) = \rho^{-m} S_m(1)$ and that the dependence of S on ρ can be cancelled by rescaling the string field as $V_q \to \rho^q V_q$. Note that all propagating on-shell string fields have zero $j_{\overline{\lambda}}$ charge, so they are unaffected by the rescaling of the regularization factor.

For closed topological strings describing Calabi-Yau three-folds, it is possible to construct a cubic closed string field theory action which resembles the action for Kodaira-Spencer gravity [7]. It would be very interesting to see if this construction for closed topological strings generalizes to the non-minimal pure spinor formalism for closed superstring field theory. Since the closed string field theory action involves the b ghost, this generalization may not be straightforward because of the singularitites in the b ghost of (3.16) when $\lambda\overline{\lambda} \to 0$. However, it is encouraging that the kinetic term for the Ramond-Ramond sector of closed superstring field theory [50] can be constructed using a set of non-minimal variables which have some similarities with the non-minimal variables of the pure spinor formalism.

6. Four-dimensional pure spinor formalism

6.1 Minimal $d = 4$ pure spinor formalism

Since topological strings are useful for computing superpotential terms in the four-dimensional spacetime action [9, 7], it is natural to look for a four-dimensional version of the pure spinor formalism. In four dimensions, the Green-Schwarz-Siegel matter variables consist of $(x^m, \theta^a, \overline{\theta}^{\dot{a}}, p_a, \overline{p}_{\dot{a}})$ for $m = 0$ to 3 and $a, \dot{a} = 1$ to 2, where p_a and $\overline{p}_{\dot{a}}$ are the conjugate momenta for θ^a and $\overline{\theta}^{\dot{a}}$. Since a $d = 4$ pure spinor is simply a chiral two-component spinor λ^a, the natural $d = 4$ version of the "minimal" pure spinor formalism is constructed from the $d = 4$ Green-Schwarz-Siegel variables, a $\hat{c} = 3$ $N = 2$ superconformal field theory for the six-dimensional compactification manifold, and a $d = 4$ pure spinor ghost λ^a together with its conjugate momentum w_a. The worldsheet action for these variables is

$$S = \int d^2z \left(\frac{1}{2}\partial x^m \overline{\partial} x_m + p_a \overline{\partial}\theta^a + \overline{p}_{\dot a}\overline{\partial}\overline{\theta}^{\dot a} - w_a \overline{\partial}\lambda^a\right) + S_C\,, \tag{6.1}$$

where S_C is the worldsheet action for the compactification-dependent variables.

The worldsheet variables in (6.1) are the same as in the $d = 4$ hybrid formalism [10] for the superstring except for the replacement of (λ^a, w_a) with a chiral boson ρ satisfying the OPE $\rho(y)\rho(z) \to -\log(y - z)$. Recall that in the $d = 4$ hybrid formalism, physical states are defined as $N = 2$ primary fields with respect to the $\hat c = 2$ $N = 2$ generators

$$\begin{aligned} J &= -\partial\rho + J_C, \quad G^+ = e^{\rho} d_a d^a + G^+_C\,, \qquad G^- = e^{-\rho}\overline{d}_{\dot a}\overline{d}^{\dot a} + G^-_C\,, \\ T &= -\frac{1}{2}\partial x^m \partial x_m - p_a\partial\theta^a - \overline{p}_{\dot a}\partial\overline{\theta}^{\dot a} - \frac{1}{2}\partial\rho\partial\rho + T_C \\ &= -\frac{1}{2}\Pi^m\Pi_m - d_a\partial\theta^a - \overline{d}_{\dot a}\partial\overline{\theta}^{\dot a} - \frac{1}{2}\partial\rho\partial\rho + T_C\,, \end{aligned} \tag{6.2}$$

where $d_a = p_a + \frac{i}{2}\partial x_m \sigma^m_{a\dot a}\overline{\theta}^{\dot a} - \frac{1}{4}(\overline{\theta})^2\partial\theta_a + \frac{1}{8}\theta_a\partial(\overline{\theta})^2$, $\overline{d}_{\dot a} = \overline{p}_{\dot a} + \frac{i}{2}\partial x_m\sigma^m_{a\dot a}\theta^a - \frac{1}{4}(\theta)^2\partial\overline{\theta}_{\dot a} + \frac{1}{8}\overline{\theta}_{\dot a}\partial(\theta)^2$, $\Pi^m = \partial x^m - \frac{i}{2}\sigma^m_{a\dot a}(\overline{\theta}^{\dot a}\partial\theta^a + \theta^a\partial\overline{\theta}^{\dot a})$, and $[J_C, G^+_C, G^-_C, T_C]$ are the $\hat c = 3$ $N = 2$ superconformal generators for the compactification manifold. After twisting, the $N = 2$ generators of (6.2) are related by a field redefinition to the RNS operators

$$J = bc + \eta\xi\,, \qquad G^+ = j^{RNS}_{BRST} \qquad G^- = b\,, \qquad T = T^{RNS}_{\text{matter}} + T^{RNS}_{\text{ghost}}\,, \tag{6.3}$$

and the $N = 2$ physical state condition is mapped to the usual requirement of BRST-invariance for RNS physical states.

In the "minimal" version of the $d = 4$ pure spinor formalism, physical states will instead be defined as ghost-number one states in the cohomology of the "minimal" BRST operator

$$Q = \int dz(\lambda^a d_a + G^+_C)\,, \tag{6.4}$$

where the ghost-number is defined by the charge

$$j_{\text{ghost}} = \int dz(w_a\lambda^a + J_C) \tag{6.5}$$

and $[J_C, G^+_C, G^-_C, T_C]$ are the twisted $\hat c = 3$ $N = 2$ superconformal generators for the compactification manifold. To compute the cohomology of Q, it is convenient to perform a similarity transformation on the worldsheet variables so that

$$d_a = p_a\,, \qquad \overline{d}_{\dot a} = \overline{p}_{\dot a} + i\partial x_m\sigma^m_{a\dot a}\theta^a - (\theta)^2\partial\overline{\theta}_{\dot a}\,, \qquad \Pi^m = \partial x^m - i\theta^a\sigma^m_{a\dot a}\partial\overline{\theta}^{\dot a}\,, \tag{6.6}$$

as in a chiral $d = 4$ superspace representation. Since states in the cohomology of $\int dz(\lambda^a p_a)$ are independent of $(\theta^a, p_a, \lambda^a, w_a)$, any ghost-number one state in the cohomology of Q can be expressed as

$$V = \Phi_j(x, \overline{\theta}, \overline{p})\psi^j\,, \tag{6.7}$$

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where Φ_j is a superfield depending on both zero modes and non-zero modes of $(x^m, \overline{\theta}^{\dot{a}}, \overline{p}_{\dot{a}})$, and ψ^j is a chiral primary of $+1$ charge with respect to the $\hat{c} = 3$ $N = 2$ superconformal field theory for the compactification manifold.

Since Φ_j can depend on the non-zero modes of $(x^m, \overline{\theta}^{\dot{a}}, \overline{p}_{\dot{a}})$, V describes both massive and massless states, and the $d = 4$ mass-shell condition is not imposed by BRST invariance. As will now be explained, V describes the chiral sector of open superstring field theory which contributes to F-terms in the open superstring field theory action. So the $d = 4$ pure spinor formalism can be understood as a $d = 4$ super-Poincaré covariant version of the $\hat{c} = 5$ topological string of [11].

When written in terms of $d = 4$ superspace variables using the hybrid formalism, the open superstring field theory action [49, 48] depends on three string fields which contain J_C charge $+1$, 0, and -1. The string field with zero J_C charge describes compactification-independent fields like the $N = 1$ $d = 4$ super-Yang-Mills multiplet, the string field with $+1$ J_C charge describes compactification-dependent fields like the chiral moduli, and the string field with -1 J_C charge describes compactification-dependent fields like the anti-chiral moduli. Although the D-term in the open superstring field theory action contains couplings between all three string fields, the F-term only involves the string field with $+1$ J_C charge which will be called V.

Using the language of the $d = 4$ hybrid formalism, V is restricted to satisfy $[\int dz G_4^+, V] = 0$ where $G_4^+ = e^\rho d_a d^a$, which implies that V has no poles with d_a. The F-term in the open superstring field theory action is given by [49, 48]

$$S = \left\langle \frac{1}{2} V \left(\int dz G_C^+ \right) V + \frac{1}{3} VVV \right\rangle_F , \tag{6.8}$$

where G_C^+ is the spin-one fermionic generator from the twisted $\hat{c} = 3$ $N = 2$ superconformal field theory for the compactification manifold, $\langle\ \ \rangle_F$ denotes the norm for F-terms defined by $\langle J_C^{+++} (\overline{\theta})^2 \rangle = 1$, and J_C^{+++} is the spectral-flow operator with $+3$ J_C charge for the $\hat{c} = 3$ $N = 2$ superconformal field theory that describes the compactification manifold.

To understand the definition of $\langle\ \ \rangle_F$, note that the norm $\langle\ \ \rangle_D$ for D-terms is defined by $\langle J_C^{+++} e^{-\rho} (\theta)^2 (\overline{\theta})^2 \rangle_D = 1$, which maps to $\langle c\partial c \partial^2 c \xi e^{-2\phi} \rangle_D = 1$ using the field redefinition to the RNS formalism. Since

$$\left[\int dz G_4^+ , J_C^{+++} e^{-\rho} (\theta)^2 (\overline{\theta})^2 \right] = J_C^{+++} (\overline{\theta})^2 , \tag{6.9}$$

one finds that $\langle [\int dz G_4^+, A] \rangle_F = \langle A \rangle_D$ for any function A, which is the superstring generalization of the usual superspace relation between F-terms and D-terms that $\langle D_a D^a A \rangle_F = \langle A \rangle_D$ where D_a are the N=1 $d = 4$ chiral superspace derivatives.

For example, for compactification on T^6 where the worldsheet variables are (y^j, ψ^j) and $(\overline{y}_j, \overline{\psi}_j)$ for $j = 1$ to 3, the twisted $\hat{c} = 3$ $N = 2$ generators for the compactification manifold are $T_C = \partial y^j \partial \overline{y}_j + \overline{\psi}_j \partial \psi^j$, $G_C^+ = \partial \overline{y}_j \psi^j$, $G_C^- = \partial y^j \overline{\psi}_j$ and $J_C = \psi^j \overline{\psi}_j$. Besides depending on chiral superfields coming from Kaluza-Klein reduction of $d = 10$ massive multiplets, the string field V depends on three chiral superfields $\Sigma_j(x, \overline{\theta}, y, \overline{y})$ which come from Kaluza-Klein reduction of the $d = 10$ massless super-Yang-Mills multiplet. The dependence of the string field on these superfields is given by $V = \Sigma_j(x, \overline{\theta}, y, \overline{y}) \psi^j$, and after plugging V

into (6.8), one obtains the expected F-term

$$S = \int d^4x \int d^6y \int d^2\overline{\theta}\epsilon^{ijk}\,\mathrm{Tr}\left(\frac{1}{2}\Sigma_i\partial_j\Sigma_k + \frac{1}{3}\Sigma_i\Sigma_j\Sigma_k\right) \tag{6.10}$$

for these superfields [51].

If one drops the D-term in the open superstring field theory action and keeps only the F-term of (6.8), physical states are described by a string field V with $+1$ J_C charge, with no poles with d_a, and which satisfies the linearized equation of motion $\{\int dzG_C^+, V\} = 0$ with the linearized gauge invariance $\delta V = [\int dzG_C^+, \Omega]$. So physical states defined with respect to (6.8) carry $+1$ J_C charge, are independent of θ^a, and are chiral primaries with respect to the $\hat{c} = 3$ $N = 2$ superconformal field theory for the compactification manifold. Since this definition of physical states coincides with the definition of physical states in the $d = 4$ pure spinor formalism, it is natural to conjecture that the $d = 4$ pure spinor formalism describes the chiral sector of superstring theory which contributes to F-terms in the superstring field theory action. Evidence for this conjecture will now be provided by computing scattering amplitudes using the non-minimal version of the $d = 4$ pure spinor formalism.

6.2 Non-minimal $d = 4$ pure spinor formalism

In analogy with the non-minimal version of the $d = 10$ pure spinor formalism, the $d = 4$ non-minimal variables will consist of a bosonic chiral spinor $\overline{\lambda}_a$ and fermionic chiral spinor r_a, with conjugate momentum $\overline{w}^a$ and s^a. Since chiral two-component spinors are automatically $d = 4$ pure spinors, there are no additional constraints on $\overline{\lambda}_a$ and r_a analogous to the $d = 10$ constraints of (3.1). Although it might seem strange that the $d = 4$ non-minimal variables have the same spacetime chirality as λ_a whereas the $d = 10$ non-minimal variables had the opposite spacetime chirality, note that in four dimensions, complex conjugation in Euclidean space does not flip the chirality of spacetime spinors. The worldsheet action including the non-minimal variables is

$$S = \int d^2z \left(\frac{1}{2}\partial x^m\overline{\partial}x_m + p_a\overline{\partial}\theta^a + \overline{p}_{\dot{a}}\overline{\partial}\overline{\theta}^{\dot{a}} - w_a\overline{\partial}\lambda^a - \overline{w}^a\overline{\partial}\overline{\lambda}_a + s^a\overline{\partial}r_a\right) + S_C\,, \tag{6.11}$$

where the barred $(\overline{\theta}^{\dot{a}}, \overline{p}_{\dot{a}})$ variables will be defined to carry dotted spinor indices while the barred $(\overline{\lambda}_a, \overline{w}^a)$ variables will carry undotted spinor indices.

In order that the non-minimal variables do not affect the cohomology, the "minimal" pure spinor BRST operator $Q = \int dz(\lambda^a d_a + G_C^+)$ will be modified to the "non-minimal" BRST operator

$$Q = \int dz\left(\lambda^a d_a + \overline{w}^a r_a + G_C^+\right). \tag{6.12}$$

It is straightforward to construct a b ghost satisfying $\{Q, b\} = T$ and one finds

$$b = s^a\partial\overline{\lambda}_a + w_a\partial\theta^a + G_C^- + \frac{i\overline{\lambda}_a\Pi^m\overline{\sigma}_m^{\dot{a}a}\overline{d}_{\dot{a}}}{2(\overline{\lambda}\lambda)} - \frac{(\epsilon^{ab}\overline{\lambda}_a r_b)(\epsilon^{\dot{a}\dot{b}}\overline{d}_{\dot{a}}\overline{d}_{\dot{b}})}{4(\overline{\lambda}\lambda)^2}\,, \tag{6.13}$$

where

$$T = -\frac{1}{2}\partial x^m \partial x_m - p_a \partial\theta^a - \overline{p}_{\dot a}\partial\overline{\theta}^{\dot a} + w_a\partial\lambda^a + \overline{w}^a\partial\overline{\lambda}_a - s^a\partial r_a + T_C \tag{6.14}$$

$$= -\frac{1}{2}\Pi^m\Pi_m - d_a\partial\theta^a - \overline{d}_{\dot a}\partial\overline{\theta}^{\dot a} + w_a\partial\lambda^a + \overline{w}^a\partial\overline{\lambda}_a - s^a\partial r_a + T_C\,, \tag{6.15}$$

and d_a, $\overline{d}_{\dot a}$ and Π^m are defined in (6.6).

One can verify that b has no poles with itself and that the double pole of b with the BRST integrand $j_{BRST} = \lambda^a d_a + \overline{w}^a r_a + G_C^+$ produces the U(1) generator

$$J = \lambda^a w_a + r_a s^a + J_C\,. \tag{6.16}$$

The generators $[J, j_{BRST}, b, T]$ form a $\hat{c} = 3$ $N = 2$ algebra which allow the formalism to be interpreted as a critical topological string. However, as in the $d = 10$ non-minimal pure spinor formalism, it is convenient to shift the U(1) generator by a BRST-trivial quantity $\{Q, -s^a\overline{\lambda}_a\} = -\overline{w}^a\overline{\lambda}_a - r_a s^a$ so that the new ghost charge is

$$j_{\text{ghost}} = \int dz J = \int dz(\lambda^a w_a + r_a s^a + J_C + \{Q, -s^a\overline{\lambda}_a\}) = \int dz(\lambda^a w_a - \overline{\lambda}_a\overline{w}^a + J_C)\,. \tag{6.17}$$

The standard topological rules for computing scattering amplitudes can now be applied using the BRST operator of (6.12), the b ghost of (6.13), the stress tensor of (6.14), and the ghost charge of (6.17). For example, N-point tree amplitudes are computed by the correlation function

$$\mathcal{A} = \langle \mathcal{N}(y) V_1(z_1) V_2(z_2) V_3(z_3) \int dz_4 U_4(z_4) \ldots \int dz_N U_N(z_N)\rangle\,, \tag{6.18}$$

where, as in (4.7), the regularization factor

$$\mathcal{N} = \exp(\{Q, -\overline{\lambda}_a\theta^a\}) = \exp(-\overline{\lambda}_a\lambda^a - r_a\theta^a) \tag{6.19}$$

will be inserted into the correlation function.

After integrating out the worldsheet non-zero modes, the zero mode integral is

$$\langle \mathcal{N}\ f(\lambda, \overline{\lambda}, r, \theta, \overline{\theta}, \psi)\rangle = \int d^2\lambda d^2\overline{\lambda} d^2 r d^2\theta d^2\overline{\theta} d^3\psi \exp(-\overline{\lambda}_a\lambda^a - r_a\theta^a) f(\lambda, \overline{\lambda}, r, \theta, \overline{\theta}, \psi)\,, \tag{6.20}$$

which is well-defined as long as $f(\lambda, \overline{\lambda}, r, \theta, \overline{\theta}, \psi)$ diverges slower than $(\lambda\overline{\lambda})^{-2}$ as $\lambda\overline{\lambda} \to 0$.

The restriction that $f(\lambda, \overline{\lambda}, r, \theta, \overline{\theta}, \psi)$ diverges slower than $(\lambda\overline{\lambda})^{-2}$ is related to the operator $\xi = (\overline{\lambda}\theta)/(\overline{\lambda}\lambda + r\theta)$ which satisfies $Q\xi = 1$. Using the same argument as in the $d = 10$ non-minimal pure spinor formalism, $\langle \mathcal{N}\ Q\Omega\rangle$ is only guaranteed to vanish if Ω diverges slower than $\lambda^{-2}\overline{\lambda}^{-1}$ as $\lambda\overline{\lambda} \to 0$. This allows $\langle \mathcal{N}\ f\rangle$ to be non-vanishing when $f = (\overline{\theta})^2(\psi)^3$ since although $\langle \mathcal{N}\ f\rangle = \langle \mathcal{N}\ Q(\xi f)\rangle$, ξf diverges like $(\theta)^2(\overline{\theta})^2(\psi)^3(\overline{\lambda}r)/(\lambda\overline{\lambda})^2$ when $\lambda\overline{\lambda} \to 0$.

Returning to the N-point tree amplitude computation, suppose that all external states are chosen in the gauge where the vertex operators are independent of the non-minimal fields. Then after integrating out the non-zero modes, one obtains

$$\mathcal{A} = \int d^2\lambda d^2\overline{\lambda} d^2 r d^2\theta d^2\overline{\theta} d^3\psi \exp(-\overline{\lambda}_a\lambda^a - r_a\theta^a)(\psi)^3 f(\overline{\theta})\,, \tag{6.21}$$

where all ghost charge in the vertex operators must come from the compactification-dependent variables ψ^j since states in the cohomology are independent of λ^a and θ^a. Integrating over λ^a, $\overline{\lambda}_a$ and r_a, one finds

$$\mathcal{A} = \int d^2\theta d^2\overline{\theta} d^3\psi \;\; (\theta)^2(\psi)^3 f(\overline{\theta}) = \int d^2\overline{\theta} f(\overline{\theta}) \,, \tag{6.22}$$

which is the desired result for the F-term in the scattering amplitude.

One can also compute N-point tree amplitudes in a worldsheet reparameterization invariant manner using $(N-3)$ b ghosts and N integrated vertex operators as

$$\mathcal{A} = \left\langle \mathcal{N}(y) V_1(z_1) \ldots V_N(z_N) \int dz_4 b(z_4) \ldots \int dz_N b(z_N) \right\rangle , \tag{6.23}$$

where $b(z)$ is defined in (6.13) and $\mathcal{N}$ is defined in (6.19). But unlike the $d = 10$ computation, there is no restriction on the number of b ghosts in the $d = 4$ computation. This is because all ghost charge in physical states must come from the compactification-dependent variables, so each unintegrated vertex operator contributes $+1$ J_C charge. By charge conservation of J_C, this implies that the only term which contributes in $b(z)$ is the G^-_C term which carries -1 J_C charge. Since G^-_C has no singularities when $\lambda\overline{\lambda} \to 0$, there is no restriction on the number of b ghosts in the $d = 4$ pure spinor formalism.

To compute N-point g-loop amplitudes, one uses the topological prescription

$$\mathcal{A} = \int d^{3g-3}\tau \left\langle \mathcal{N}(y) \prod_{j=1}^{3g-3} \left(\int dw_j \mu_j(w_j) b(w_j) \right) \prod_{r=1}^{N} \int dz_r \, \mathrm{U}(z_r) \right\rangle , \tag{6.24}$$

where τ_j are the complex Teichmuller parameters and μ_j are the associated Beltrami differentials,

$$\mathcal{N}(y) = \exp(\{Q, \chi(y)\}) = \exp\left(-\overline{\lambda}_a(y)\lambda^a(y) - r_a(y)\theta^a(y) - \sum_{I=1}^{g} (w^I_a \overline{w}^{aI} - d^I_a s^{aI}) \right) , \tag{6.25}$$

$\chi(y) = -\overline{\lambda}_a(y)\theta^a(y) - \sum_{I=1}^{g} w^I_a s^{aI}$, and $(w^I_a, \overline{w}^{aI}, s^{aI}, d^I_a, \overline{d}^I_{\dot a}, \overline{\psi}^I_j)$ for $I = 1$ to g are the zero modes for the variables of $+1$ conformal weight.

After separating off the zero modes of $(w_a, \overline{w}^a, s^a, d_a, \overline{d}_{\dot a}, \overline{\psi}_j)$ as in (4.12) and integrating over the non-zero modes, one obtains

$$\begin{aligned}\mathcal{A} = & \int d^2\lambda d^2\overline{\lambda} d^2 r d^2\theta d^2\overline{\theta} d^3\psi \prod_{I=1}^{g} d^2 w^I d^2\overline{w}^I d^2 s^I d^2 d^I d^2\overline{d}^I d^3\overline{\psi}^I \times \\ & \times \mathcal{N} f(\lambda, \overline{\lambda}, r, \theta, \overline{\theta}, \psi, w^I, \overline{w}^I, d^I, \overline{d}^I, s^I, \overline{\psi}^I) \,, \end{aligned} \tag{6.26}$$

where f is some BRST-invariant function of the zero modes with U(1) charge $3-3g$. Since conservation of J_C charge implies that only the G^-_C term in the b ghost contributes to the g-loop amplitude, f has no singularities when $\lambda\overline{\lambda} \to 0$ and there is no restriction on the number of b ghosts or on the genus g.

One can easily check that this prescription for the closed superstring reproduces the g-loop scattering amplitude of $2g$ self-dual graviphotons and an arbitrary number of chiral superfields for the Calabi-Yau moduli [9, 7]. As in the computation using the $d = 4$ hybrid formalism [22] or using the $\hat{c} = 5$ topological formalism [11], the $2g$ zero modes for $\overline{d}_{\dot{a}}$ come from the graviphoton vertex operators, the $3g - 3$ zero modes for $\overline{\psi}_j$ come from the b ghosts, and the two $\overline{\theta}^{\dot{a}}$ zero modes come from the Calabi-Yau chiral superfields. The remaining $2g$ zero modes for d_a, $2g$ zero modes for s^a, two zero modes for r_a, and two zero modes for θ^a come from the regularization factor $\mathcal{N}$ of (6.25).

So the topological string prescription for scattering amplitudes using the $d = 4$ pure spinor formalism correctly reproduces the F-term in the spacetime action. Further confirmation that the $d = 4$ pure spinor formalism describes F-terms comes from the open string field theory action for the $d = 4$ pure spinor formalism. Using the construction of section 4, the open string field theory action for the $d = 4$ pure spinor formalism is

$$S = \left\langle \left(\frac{1}{2} VQV + \frac{1}{3} VVV \right) \mathcal{N} \left(\frac{\pi}{2} \right) \right\rangle , \tag{6.27}$$

where Q is defined in (6.12) and $\mathcal{N}$ is defined in (6.19). The action of (6.27) has the same Chern-Simons structure as the F-term of (6.8) in the open superstring field theory action, and it should not be difficult to prove their equivalence. It would be interesting to generalize this construction of the F-term in non-trivial closed string backgrounds involving Ramond-Ramond fields.

A. U(5)-covariant variables for the non-minimal formalism

In this appendix, the constraints of (2.1) and (3.1) for the pure spinor ghost and non-minimal variables will be solved in a U(5)-covariant manner in terms of free fields. The coefficients in the OPE's of (2.6) and (3.7) can then be computed using the free field OPE's of the U(5)-covariant variables.

As shown in [1], the pure spinor constraint $\lambda \gamma^m \lambda = 0$ can be solved in terms of free fields as

$$\lambda^\alpha = (\lambda^+, \lambda_{ab}, \lambda^a) = \left(\gamma, \gamma u_{ab}, -\frac{1}{8} \gamma \epsilon^{abcde} u_{bc} u_{de} \right) , \tag{A.1}$$

where $a = 1$ to 5, $u_{ab} = -u_{ba}$, and $(\lambda^+, \lambda_{ab}, \lambda^a)$ describe the $(1, 10, \overline{5})$ components of λ^α under the U(5) decomposition of the (Wick-rotated) SO(10) pure spinor. In terms of the variables (γ, u_{ab}) and their conjugate momenta (β, v^{ab}), the gauge-invariant currents of (2.3) are

$$\begin{aligned}
N^{ab} &= v^{ab} , \\
N_a^b &= -u_{ac} v^{bc} + \delta_a^b \left(\frac{5}{4} \eta \xi + \frac{3}{4} \partial \phi \right) , \\
N_{ab} &= 3 \partial u_{ab} + u_{ac} u_{bd} v^{cd} - u_{ab} \left(\frac{5}{2} \eta \xi + \frac{3}{2} \partial \phi \right) , \\
J_\lambda &= -\frac{5}{2} \partial \phi - \frac{3}{2} \eta \xi , \\
T_\lambda &= \frac{1}{2} v^{ab} \partial u_{ab} - \eta \partial \xi - \frac{1}{2} (\partial \phi \partial \phi + \partial^2 \phi) - \frac{7}{2} \partial (\eta \xi + \partial \phi) ,
\end{aligned} \tag{A.2}$$

where $\gamma = \eta e^{\phi}$ and $\beta = \partial\xi e^{-\phi}$. It is straightforward to use the free field OPE's

$$v^{ab}(y)u_{cd}(z) \to \delta_c^{[a}\delta_d^{b]}(y-z)^{-1}\,, \quad \eta(y)\xi(z) \to (y-z)^{-1}\,, \quad \phi(y)\phi(z) \to -\log(y-z)\,, \quad \text{(A.3)}$$

to show that these currents satisfy the OPE's of (2.6).

To describe the non-minimal variables $(\overline{\lambda}_\alpha, r_\alpha)$ in terms of unconstrained U(5)-covariant variables, define

$$\begin{aligned}
\overline{\lambda}_\alpha &= \left(\overline{\lambda}_+, \overline{\lambda}^{ab}, \overline{\lambda}_a\right) = \overline{\gamma}\left(1, \overline{u}^{ab}, -\frac{1}{8}\epsilon_{abcde}\overline{u}^{bc}\overline{u}^{de}\right), \\
r_\alpha &= (r_+, r^{ab}, r_a) = \overline{\gamma}\left(f, f^{ab} + f u^{ab}, -\frac{1}{8}\epsilon_{abcde}\left(f\overline{u}^{bc}\overline{u}^{de} + 2f^{bc}\overline{u}^{de}\right)\right), \qquad \text{(A.4)}
\end{aligned}$$

which satisfy the constraints $\overline{\lambda}\gamma^m\overline{\lambda} = r\gamma^m\overline{\lambda} = 0$ of (3.1). In terms of the variables $(\overline{\gamma}, \overline{u}^{ab}, f, f^{ab})$ and their conjugate momenta $(\overline{\beta}, \overline{v}_{ab}, g, g_{ab})$, the gauge-invariant currents of (3.4) and (3.6) are

$$\begin{aligned}
\overline{N}_{ab} &= \overline{v}_{ab}\,, \\
\overline{N}_a^{\,b} &= \overline{u}^{bc}\overline{v}_{ac} + f^{bc}g_{ac} + \delta_a^b\left(-\frac{1}{4}\overline{\eta}\overline{\xi} + \frac{1}{4}\partial\overline{\phi}\right), \\
\overline{N}^{ab} &= \overline{u}^{ac}\overline{u}^{bd}\overline{v}_{cd} + \overline{u}^{ac}f^{bd}g_{cd} + f^{ac}\overline{u}^{bd}g_{cd} - f^{ab}g - \overline{u}^{ab}\left(\frac{1}{2}\overline{\eta}\overline{\xi} - \frac{1}{2}\partial\overline{\phi}\right), \\
\overline{J}_{\overline{\lambda}} &= -\frac{1}{2}\partial\overline{\phi} + \frac{1}{2}\overline{\eta}\overline{\xi}\,, \quad J_r = fg + \frac{1}{2}f^{ab}g_{ab} + 8(\overline{\eta}\overline{\xi} + \partial\overline{\phi})\,, \quad \Phi = \frac{1}{2}f^{ab}\overline{v}_{ab} + \frac{1}{2}f(\overline{\eta}\overline{\xi} - \partial\overline{\phi})\,, \\
T_{\overline{\lambda}} &= \frac{1}{2}\overline{v}_{ab}\partial\overline{u}^{ab} - \frac{1}{2}g_{ab}\partial f^{ab} - g\partial f - \frac{1}{2}(\overline{\eta}\partial\overline{\xi} + \overline{\xi}\partial\overline{\eta}) - \frac{1}{2}\partial\overline{\phi}\partial\overline{\phi}\,, \\
S &= g, \quad S^{ab} = \overline{u}^{ac}\overline{u}^{bd}g_{cd} - \overline{u}^{ab}g\,, \quad S_a^b = \overline{u}^{bc}g_{ac} - \frac{1}{2}\delta_a^b g\,, \quad S_{ab} = g_{ab}\,, \qquad \text{(A.5)}
\end{aligned}$$

where $\overline{\gamma} = \overline{\eta}e^{\overline{\phi}}$ and $\overline{\beta} = \partial\overline{\xi}e^{-\overline{\phi}}$. It is straightforward to use the free field OPE's

$$\begin{aligned}
&\overline{v}_{ab}(y)\overline{u}^{cd}(z) \to \delta_{[a}^c\delta_{b]}^d(y-z)^{-1}\,, \qquad \overline{\eta}(y)\overline{\xi}(z) \to (y-z)^{-1}\,, \qquad \overline{\phi}(y)\overline{\phi}(z) \to -\log(y-z)\,, \\
&\quad g(y)f(z) \to (y-z)^{-1}\,, \qquad g_{ab}(y)f^{cd}(z) \to \delta_{[a}^c\delta_{b]}^d(y-z)^{-1}\,, \qquad \text{(A.6)}
\end{aligned}$$

to show that these currents satisfy the OPE's of (3.7).

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